A definition of self-adjoint operators derived from the Schrödinger operator with the white noise potential on the plane

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https://www.math.h.kyoto-u.ac.jp/users/ueki/presen202509MS 2DWN-Scr-c1.pdf

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Motivation

Motivation: Realize $-\Delta + \xi$ as a self-adjoint operator on $L^2(\mathbb{R}^d)$
to apply the spectral analysis

White noise: $\xi = (\xi(x))_{x \in \mathbb{R}^d} \underset{\text{i.i.d.}}{\sim} \mathrm{N}(0, *)$: The most basic but wild random field

ξ: Gaussian random field, $\mathbb{E}[\xi(x)] = 0$, $\mathbb{E}[\xi(x)\xi(y)] = \delta(x - y)$

Difficulty: "$x \mapsto \xi(x)$" $\in C^{-\epsilon - d/2}$, not a regular function

Dimensions

$$d = 1 \Rightarrow W^{1,2}([a,b]) \ni f, g \mapsto \int_a^b (f'g' + \underset{C_{loc}^{-1/2-\epsilon}}{\xi} fg)dx : \text{ well-defined}$$

$-\Delta + \xi$ is realized as a self-adjoint operator

M. Fukushima and S. Nakao (1977) Spectral asymptotics
(Asymptotics of the Integrated density of states)

N. Minami (1988) (1989) Anderson Localization at all energies

L. Dumaz and C. Labbé (2020) (2023) Eigenvalues Eigenvectors Statistics

Recently

$d = 2$ or $3 \Rightarrow -\Delta + \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) + c_\varepsilon)$ are realized as self-adjoint operators

$d \geq 4 \Rightarrow$ No results

Related works on singular SPDEs

M. Hairer (2014) A theory of regularity structures,
M. Gubinelli, P. Imkeller and N. Perkowski (2015) Paracontrolled calculus
A. Kupiainen (2016) Renormalization Group
$\Rightarrow$ Stochastic quatization equation for ϕ^4_3 Euclidean quantum field theory
Generalized continuous parabolic Anderson model
Kardar–Parisi–Zhang type equation
Navier-Stokes equation with very singular forcing
and so on

Eg. Continuous parabolic Anderson model

$$\partial_t u(t,x) = \Delta_x u(t,x) - \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) + c_\varepsilon)u(t,x) \text{ for } t > 0$$

Schrödinger operators on compact spaces

R. Allez and K. Chouk (2015) Paracontrolled calculus based on Fourier Analysis

$-\Delta + \lim_{\varepsilon \to 0} (\xi_\varepsilon(x) + c_\varepsilon)$ on $\mathbb{R}^2/\mathbb{Z}^2$: Self-adjoint, Norm resolvent limit of C^∞-app.

↑Fourier partial sum

(Discrete Spectrum, Asymptotic Distribution)

M. Gubinelli, B. Ugurcan and I. Zachhuber (2020) Extension to $\mathbb{R}^3/\mathbb{Z}^3$

$$i\partial_t u(t,x) = \underbrace{(\Delta - \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) + c_\varepsilon) - k_\xi)}_{\geq 0} u(t,x) - (u|u|^2)(t,x),\ \underset{\text{given}}{u(0,\cdot)}$$

$$\partial_t^2 u(t,x) = (\Delta - \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) + c_\varepsilon) - k_\xi) u(t,x) - (u^3)(t,x),\ \underset{\text{given}}{(u(0,\cdot), \partial_t u(0,\cdot))}$$

Well-posedness,

The convergence of the solutions of regularized equations

C. Labbé (2019) similar results by the theory of regularity structure for

$-\Delta + \lim_{\varepsilon \to 0} (\xi_\varepsilon(x) + c_\varepsilon)$ on $(-L, L)^{2or3}$ with periodic or Dirichlet conditions

↑convolution with C_0^∞ function

Topics on Random Schrödinger operators

Anderson transition:

Point spectrum with exponentially decaying eigenstates
for strongly random potentials and energies near the edge of the spectrum (Anderson localization)

Absolutely continuous spectrum
for weakly random potentials and energies far from the edge of the spectrum

This topic is discussed for stationary potentials on noncompact spaces.

Extensions to noncompact spaces

M. Hairer and C. Labbé, (2015) (2018)

$\partial_t u = \Delta u - \lim_{\varepsilon \to 0} (\xi_\varepsilon + c_\varepsilon) u, t > 0, x \in \mathbb{R}^d, \ d = 2, 3, \ u(0, \cdot)$

↑convolution with C_0^∞ function — given

Well-posedness,

The convergence of the solutions of regularized equations

Y. Hsu and C. Labbé, (2024)

Construct a self adjoint operator $-\Delta + \lim_{\varepsilon \to 0} (\xi_\varepsilon + c_\varepsilon)$ on $\mathbb{R}^d$, $d = 2, 3$,

↑convolution with C_0^∞ function

as the generator of the parabolic Anderson model

For the operator, Spec$= \mathbb{R}$

Related operators on noncompact spaces

B. Ugurcan, (2022) $-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) + \widetilde{c}_\varepsilon(x))$ on $\mathbb{R}^2$ with $\widetilde{c}_\varepsilon(x) \xrightarrow{|x|\to\infty} 0$

$$= \underline{-\Delta + \lim_{\varepsilon\to 0}(\xi^{\uparrow}_\varepsilon(x) + \widetilde{c}_\varepsilon(x))} + \underline{\xi^{\downarrow}_\varepsilon(x)},$$ extension of GUZ(2020)

$\uparrow$ An extension of the method for the compact case

$\nwarrow$ By a commutator estimate,

where $\xi = \xi^{\uparrow}(x) + \xi^{\downarrow}(x)$ (Smooth functions in x)

$$= \sum_{n=-1}^{\infty} \widetilde{\chi_{[[2^n],2^{n+1}]}}(|x|)\{\underbrace{\widetilde{\chi_{[c2^n,\infty)}}(-\Delta)\xi}_{\text{high energy pert}} + \underbrace{\widetilde{\chi_{[0,c2^n]}}(-\Delta)\xi}_{\text{low energy pert}}\}$$

$\widetilde{c}_\varepsilon(x) = \mathbb{E}[$A resonant product of $\xi^{\uparrow}_\varepsilon(x)$ and $(1-\Delta)^{-1}\xi^{\uparrow}_\varepsilon(x)]$

Heat semigroup approach in the paracontrolled calculus

Use the heat semigroup to multiply functions

(cf. the more traditional approach uses Fourier analysis.)

(Applicable to many kind of configuration spaces as manifolds, graphs,....)

I. Bailleul and F. Bernicot (2016) For generalized PAM on $\underbrace{\text{2D manifold}}_{\text{without compactness}}$

Well-posedness, The convergence of the solutions of regularized equations

I. Bailleul, F. Bernicot and D. Frey (2018)

For PAM and multiplicative Burgers eq. on $\underbrace{\text{3D manifold}}_{\text{without compactness}}$

Well-posedness, The convergence of the solutions of regularized equations

A. Mouzard (2022) Self-adjointness, Norm-resolvent limit of C^∞-app. for

$\underbrace{-\Delta}_{\text{eg. Lap.Belt}} + \lim_{\varepsilon \to 0}(\underbrace{\xi_\varepsilon(x)}_{\text{e.g. } (e^{\varepsilon^2 \Delta}\xi)(x)} + c_\varepsilon(x))$ on 2D compact manifold.

$$c_\varepsilon(x) \equiv c_\varepsilon \text{ on } \mathbb{R}^2/\mathbb{Z}^2$$

Our Topics

By the paracontrolled calcuclus by the heat semigroup referring Mouzard (2020)
and the partition of unity,

$$\widetilde{H^\xi} = -\Delta + \lim_{\varepsilon\to 0}\Big(\underset{(e^{\varepsilon^2\Delta}\xi)(x)}{\underset{\|}{\xi_\varepsilon(x)}} + c_\varepsilon \Big) \text{ on } \mathbb{R}^2.$$

Self-adjointness.

$\mathrm{Spec}(\widetilde{H^\xi}) = \mathbb{R}$ as in Hsu and Labbé (2024).

$\rightsquigarrow$ In progress

Anderson localization at sufficiently low energies by a traditional proof.

https://www.math.h.kyoto-u.ac.jp/users/ueki/2DWN-Loc1.pdf

Comparison between Hsu-Labbe's definition and our definition

Advantages of Hsu-Labbe's definition

- ▷ The 3-dimensional case is included.
- ▷ The expressions are simpler.
- ▷ $\xi_\varepsilon(x) = \xi * \rho(\cdot/\varepsilon)/\varepsilon^d$ with a general C_0^∞ function ρ.

  (cf. $\xi_\varepsilon(x) = (e^{\varepsilon^2\Delta}\xi)(x)$ basically in the heat semigroup approach.)

Advantages of our definition

- ▷ More convenient for arguments that handle the operator directly.

  Examples: ∘ Our proof of "$\mathrm{Spec}(\widetilde{H^\xi}) = \mathbb{R}$" approaches $\widetilde{H^\xi}$ directly.

  ∘ The most used proof of the Anderson localization is a combination of several results, each of which is valuable in its own right.

  The proof may be better suited to our definition.

Products $fg = P_f g + \Pi(f,g) + P_g f + P_1^{(b)}((P_1^{(b)} f)(P_1^{(b)} g))$

$0 << b \in 2\mathbb{Z}$ fixed $\quad P_t^{(b)} = \sum_{j=0}^{b-1} \frac{(-t\Delta)^j}{j!} e^{t\Delta}$

Standard families of Gaussian operators with cancellation of orders $[s_1, s_2]$

$StGC^{[s_1,s_2]} = \{(Q_t = (\sqrt{t}\nabla)^\alpha P_t^{(c)})_{t\in(0,1]} : \alpha \in \mathbb{Z}_+^2, s_1 \le \alpha_1 + \alpha_2 \le s_2, 1 \le c \le b\}$

$|Q_t(x,y)| \lesssim \exp(-c|x-y|^2/t)/t$

$$\underbrace{P_0^{(b)}((P_0^{(b)} f)(P_0^{(b)} g))}_{=fg} - P_1^{(b)}((P_1^{(b)} f)(P_1^{(b)} g)) = -\int_0^1 dt \partial_t \{P_t^{(b)}((P_t^{(b)} f)(P_t^{(b)} g))\}$$

$$= P_f g + \Pi(f,g) + P_g f$$

$P_f g := \sum_\nu c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g))$: paraproduct term, $\exists$distribution

$\Pi(f,g) := \sum_\mu c_\mu \int_0^1 \frac{dt}{t} P_t^\mu((Q_t^{1,\mu} f)(Q_t^{2,\mu} g))$: resonating term, require regularity

$P^\nu, P^\mu \in StGC^{[0,b/2)}$, $Q^{1,\nu}, Q^{2,\nu}, Q^{1,\mu}, Q_t^{2,\mu} \in StGC^{[b/2,2b]}$

The Besov Spaces

For $p, q \in [1, \infty], \alpha \in (-2b, 2b)$, $\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2) = \overline{C_0^{\infty}(\mathbb{R}^2)}^{\|\cdot\|_{\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)}}$: the Besov Space

$$\|f\|_{\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)} := \|e^{\Delta} f\|_{L^p(\mathbb{R}^2:dx)} + \sup\{\|t^{-\alpha/2}\|Q_t f\|_{L^p(\mathbb{R}^2:dx)}\|_{L^q([0,1]:t^{-1}dt)} : Q \in StGC^{(|\alpha|,2b]}\}$$

$\mathcal{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^2) =: \mathcal{C}^{\alpha}(\mathbb{R}^2)$: the α-Hölder space

$\mathcal{B}^{\alpha}_{2,2}(\mathbb{R}^2) =: \mathcal{H}^{\alpha}(\mathbb{R}^2)$: the Sobolev space with the index α.

$\{\chi_a\}_{a\in\mathbb{Z}^2} \subset C^{\infty}(\mathbb{R}^2 \to [0,1])$ s.t. $\sum_{a\in\mathbb{Z}^2} \chi_a^2 \equiv 1$, $\operatorname{supp}\chi_a \subset \square_2(a) := a + (-1,1)^2$

$\chi_a(\cdot) = \chi_0(\cdot - a)$

The continuity of paraproduct and resonating terms

(i) For any $\alpha, \beta \in \mathbb{R}$ and $\epsilon \in (0,1)$,
$\|\chi_{a_1} P_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{H}^{(\alpha\wedge 0)+\beta-\epsilon}(\mathbb{R}^2)}$

$$\leq \begin{cases} C_{\alpha,\beta,\epsilon}\|\chi_{a_2} f\|_{\mathcal{C}^{\alpha\wedge 0}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{H}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \\ C_{\alpha,\beta,\epsilon}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha\wedge 0}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \end{cases}$$

(ii) For any $\alpha, \beta \in \mathbb{R}$ such that $\alpha+\beta > 0$,
$\|\chi_{a_1} \Pi(\chi_{a_2} f, \chi_{a_3} g)\|_{\mathcal{H}^{\alpha+\beta}(\mathbb{R}^2)}$
$\leq C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2))$.

1st ansatz for the definition of the operator

$\|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \leq C_{\epsilon,\xi}(\log(2+|a|))^{1/2}$

$u, H^\xi u := -\Delta u + \xi u \in L^2(\mathbb{R}^2) \Rightarrow \Delta u \in \mathcal{H}^{-1-\epsilon}(\mathbb{R}^2) \Rightarrow u \in \mathcal{H}^{1-\epsilon}(\mathbb{R}^2)$

$$H^\xi u = \underbrace{-\Delta u}_{\in \mathcal{H}_{loc}^{-1-\epsilon}\uparrow} + \underbrace{P_u \xi}_{\in \mathcal{H}_{loc}^{-1-\epsilon}\uparrow} + \underbrace{P_\xi u}_{\in \mathcal{H}_{loc}^{-2\epsilon}} + \underbrace{\Pi(u,\xi)}_{\text{ill defned}} + (L^2)$$

To erase this singularity, $-\Delta u + P_u \xi = -\underbrace{\Delta \phi_\xi(u)}_{\in \mathcal{H}_{loc}^{-2\epsilon}} + (L^2)$

Ansatz I: $u = \Delta^{-loc} P_u \xi + \phi_\xi(u)^{\nearrow}$ with $\phi_\xi(u) \in \mathcal{H}^{2(1-\epsilon)}(\mathbb{R}^2)$,
and $\|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}, \|\chi_a \phi_\xi(u)\|_{\mathcal{H}^{2(1-\epsilon)}(\mathbb{R}^2)}$ decays sufficiently fast as $|a| \to \infty$

where $\Delta^{-loc} := -\int_0^1 dt\ e^{t\Delta}$ satisfying $\Delta^{-loc}\Delta = \Delta\Delta^{-loc} = I - e^{\Delta}$

$\|\chi_{a_1}\Delta^{-loc}\chi_{a_2} f\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} \leq C_{\alpha,\epsilon}\|\chi_{a_2} f\|_{\mathcal{C}^{\alpha+\epsilon-2}(\mathbb{R}^2)} \exp(-C|a_1 - a_2|^2)$

$\|\chi_{a_1}\Delta^{-loc}\chi_{a_2} f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \leq C_{\alpha,\epsilon}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha+\epsilon-2}(\mathbb{R}^2)} \exp(-C|a_1 - a_2|^2)$

Commutators and Products of 3 functions

$$H^\xi u = -\underbrace{\Delta\phi_\xi(u)}_{\in\mathcal{H}^{-2\epsilon}} + \underbrace{P_\xi(\Delta^{-loc}P_u\xi)}_{\in\mathcal{H}^{-3\epsilon}} + \underbrace{\Pi(\Delta^{-loc}P_u\xi,\xi)}_{\text{ill defned}} + (L^2)$$

In the 2nd and 3rd terms, move the function u to outer places by

$C(f,g,h) := \Pi(\Delta^{-loc}P_f g, h) - f\Pi(\Delta^{-loc}g, h)$

$S(f,g,h) := P_h(\Delta^{-loc}P_f g) - {}_fP_h(\Delta^{-loc}g)$

where ${}_fP_h g := \sum_\nu c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu h)(Q_t^{2,\nu}g)f)$

(i) For any $\epsilon,\alpha\in(0,1), \beta\in\mathbb{R}, \gamma\in(-\infty,0)$ such that $\beta+\gamma<0$ and $\alpha+\beta+\gamma>0$,

$\|\chi_{a_1}C(\chi_{a_2}f,\chi_{a_3}g,\chi_{a_4}h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)}, \|\chi_{a_1}S(\chi_{a_2}f,\chi_{a_3}g,\chi_{a_4}h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)}$

$\leq C_{\epsilon,\alpha,\beta,\gamma}\|\chi_{a_2}f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)}\|\chi_{a_3}g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)}\|\chi_{a_4}h\|_{\mathcal{C}^\gamma(\mathbb{R}^2)}$

$\quad\times\exp(-C(|a_1-a_2|^2+|a_1-a_3|^2+|a_1-a_4|^2))$

(ii) For any $\alpha\in(-\infty,0)$, $\beta\in\mathbb{R}$ and $\epsilon\in(0,1)$,

$\|\chi_{a_1}\,{}_{\chi_{a_2}h}P_{\chi_{a_3}f}(\chi_{a_4}g)\|_{\mathcal{H}^{\alpha+\beta-\epsilon}(\mathbb{R}^2)}$

$\leq C_{\alpha,\beta,\epsilon}\|\chi_{a_3}f\|_{\mathcal{C}^\alpha(\mathbb{R}^2)}\|\chi_{a_4}g\|_{\mathcal{C}^\beta(\mathbb{R}^2)}\|\chi_{a_2}h\|_{L^2(\mathbb{R}^2)}$

$\quad\times\exp(-C(|a_1-a_2|^2+|a_2-a_3|^2+|a_2-a_4|^2))$

Modification

Then $H^\xi u = -\underbrace{\Delta\phi_\xi(u)}_{\in\mathcal{H}^{-2\epsilon}} + \underbrace{{}_uP_\xi(\Delta^{-loc}\xi)}_{\in\mathcal{H}^{-4\epsilon}} + u\underbrace{\Pi(\Delta^{-loc}\xi,\xi)}_{\text{ill defned}} + (L^2)$

Replace the ill defned term by a $\bigcap_{\epsilon>0} C^{-\epsilon}_{loc}(\mathbb{R}^2)$-valued random variable Y_ξ s.t. $\lim_{\varepsilon\to 0}\mathbb{E}[\|\chi_a(Y_{\xi_\varepsilon} - Y_\xi)\|^p_{C^{-\epsilon}(\mathbb{R}^2)}] = 0$ for any $a \in \mathbb{Z}^2$, $p \in [1,\infty)$ and $\epsilon > 0$, where

$$Y_{\xi_\varepsilon} := \Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon) - \underline{\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon,\xi_\varepsilon)]}$$

$\xi_\varepsilon := e^{\varepsilon^2\Delta}\xi$ is a smooth approximation of ξ $\quad\nwarrow$ diverge as $\varepsilon \to 0$

$\|\chi_a Y_\xi\|_{C^{-\epsilon}(\mathbb{R}^2)} \le C_{\epsilon,\xi}\log(2+|a|)$

Here the operator H^ξ is replaced by a new operator which we denote as $\widetilde{H^\xi}$

2nd ansatz

$$\widetilde{H^{\xi}}u = -\underbrace{\Delta\phi_{\xi}(u)}_{\in\mathcal{H}^{-2\epsilon}} + \underbrace{{}_uP_{\xi}(\Delta^{-loc}\xi)}_{\in\mathcal{H}^{-4\epsilon}} + \underbrace{P_uY_{\xi}}_{\in\mathcal{H}^{-2\epsilon}} + (L^2)$$

To erase remaining singularities,

Ansatz II: $\phi_{\xi}(u) = \Delta^{-loc}({}_uP_{\xi}(\Delta^{-loc}\xi) + P_uY_{\xi}) + \Phi_{\xi}(u)$

with $\Phi_{\xi}(u) \in \mathcal{H}^2(\mathbb{R}^2)$ and $\|\chi_a\Phi_{\xi}(u)\|_{\mathcal{H}^2(\mathbb{R}^2)}$ decays sufficiently fast as $|a| \to \infty$

$\Rightarrow -\Delta\phi_{\xi}(u) +_u P_{\xi}(\Delta^{-loc}\xi) + P_uY_{\xi} = e^{\Delta}_u P_{\xi}(\Delta^{-loc}\xi) + e^{\Delta}P_uY_{\xi} - \Delta\Phi_{\xi}(u) \in L^2$,

$\therefore \widetilde{H^{\xi}}u \in L^2$

Our operator

$$
\begin{aligned}
\widetilde{H^\xi} u := & -\Delta\Phi_\xi(u) + P_\xi \Phi_\xi(u) + \Pi(\Phi_\xi(u), \xi) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} \xi)) \\
& + e^\Delta P_u \xi + e^\Delta{}_u P_\xi(\Delta^{-loc}\xi) + e^\Delta P_u Y_\xi \\
& + C(u,\xi,\xi) + S(u,\xi,\xi) \\
& + P_{Y_\xi} u + \Pi(u, Y_\xi) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} Y_\xi)) \\
& + P_\xi(\Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi)) + \Pi(\Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi), \xi) \\
& + P_\xi(\Delta^{-loc} P_u Y_\xi) + \Pi(\Delta^{-loc} P_u Y_\xi, \xi),
\end{aligned}
$$

with $\Phi_\xi(u) := u - \Delta^{-loc} P_u \xi - \Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi) - \Delta^{-loc} P_u Y_\xi$

Main Statements

$$\mathrm{Dom}_{+0}(\widetilde{H^\xi}) := \Big\{ u \in \bigcap_{\epsilon>0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} < 0 \text{ for any } \epsilon > 0,$$

$$\Phi_\xi(u) \in \mathcal{H}^2(\mathbb{R}^2), \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a \Phi_\xi(u)\|_{\mathcal{H}^2(\mathbb{R}^2)} < 0 \Big\}$$

Theorem (Self-adjointness)

The operator $\widetilde{H^\xi}$ with the domain $\mathrm{Dom}_{+0}(\widetilde{H^\xi})$ is essentially self-adjoint on $L^2(\mathbb{R}^2)$.

Theorem (Spectrum)

The spectral set of the closure $\overline{\widetilde{H^\xi}}$ is $\mathbb{R}$.

An abstract representation of the operator $\widetilde{H^\xi}$

$$
\begin{aligned}
\widetilde{H^\xi} u \sim -\Delta\Phi_\xi(u) + \sum \int_0^1 \frac{dt}{t} \int dx_1\, O\Big(\frac{1}{t}\exp\Big(-\frac{|x-x_1|^2}{t}\Big)\Big) u(x_1) \\
\times \int dx_2\, O\Big(\frac{1}{t}\exp\Big(-\frac{|x-x_2|^2}{t}\Big)\Big) \begin{matrix}\xi(x_2) \\ Y_\xi(x_2)\end{matrix} \\
\times \Big(\int dx_3\, O\Big(\frac{1}{t}\exp\Big(-\frac{|x-x_3|^2}{t}\Big)\Big)\xi(x_3)\Big)
\end{aligned}
$$

A useful tool to treat the operator $\widetilde{H^\xi}$

Smooth approximation $\widetilde{H^{\xi_\varepsilon}} = -\Delta + \xi_\varepsilon - \mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon)]$
is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ since $|\underbrace{\xi_\varepsilon(x)}_{\text{smooth}}| \leq C_{\xi,\varepsilon}(\log(2+|x|))^{1/2}$

But $C_0^\infty(\mathbb{R}^2) \not\subset \mathrm{Dom}_{+0}(\widetilde{H^\xi})$ since $\Phi_\xi(C_0^\infty(\mathbb{R}^2)) \not\subset \mathcal{H}^2(\mathbb{R}^2)$

$\mathrm{Dom}_{+0}(\widetilde{H^\xi})$ depends on ξ

Our useful tool is

$$\Phi_\xi^s(u) := u - \sum_{a\in\mathbb{Z}^2} \Delta^{-loc} P_u^{s(a)}(\chi_a^2\xi) - \sum_{a,a'\in\mathbb{Z}^2} \Delta^{-loc}\, {}_u P_{\chi_a^2\xi}^{s_1(a,a')}(\Delta^{-loc}\chi_{a'}^2\xi)$$
$$- \sum_{a\in\mathbb{Z}^2} \Delta^{-loc} P_u^{s_2(a)}(\chi_a^2 Y_\xi)$$

$$P_f^s g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g))$$

$$_h P_f^s g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g)h)$$

Choice of s

For any $\epsilon \in (0,1)$ and almost all ξ,
$\boldsymbol{s}(\epsilon,\xi,\delta) = (s(a;\epsilon,\xi,\delta), s_1(a,a';\epsilon,\xi,\delta), s_2(a;\epsilon,\xi,\delta))_{a\in\mathbb{Z}^2}$ is taken so that

$$\|\chi_a(I - \Phi_\xi^{\boldsymbol{s}(\epsilon,\xi,\delta)})(u)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \leq \delta \sum_{a'\in\mathbb{Z}^2} \exp(-M|a-a'|^2)\|\chi_{a'}u\|_{L^2(\mathbb{R}^2)}$$

Indeed, there exist $s(\epsilon,\xi), s_1(\epsilon,\xi), s_2(\epsilon,\xi) \in (0,1)$ and
$M, M(\epsilon), M_1(\epsilon), M_2(\epsilon) \in (0,\infty)$ s.t.

$$s(a;\epsilon,\xi,\delta) = s(\epsilon,\xi)\left(\frac{\delta}{(\log(2+|a|))^2}\right)^{M(\epsilon)},$$

$$s_1(a,a';\epsilon,\xi,\delta) = s_1(\epsilon,\xi)\left(\frac{\delta}{(\log(2+|a|))^2(\log(2+|a'|))^2}\right)^{M_1(\epsilon)}$$

$$s_2(a;\epsilon,\xi,\delta) = s_2(\epsilon,\xi)\left(\frac{\delta}{\log(2+|a|)}\right)^{M_2(\epsilon)}.$$

Inverse of Φ_ξ^s

$$\|(I - \Phi_\xi^{s(\epsilon,\xi,\delta)})(u)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \leq C_{\xi,\epsilon}\delta\|u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}$$

Thus for $\delta \in (0, 1/C_{\xi,\epsilon})$, there exists the inverse $(\Phi_\xi^{s(\epsilon,\xi,\delta)})^{-1} = \sum_{n=0}^{\infty}(I - \Phi_\xi^{s(\epsilon,\xi,\delta)})^n$

s.t. $\|(\Phi_\xi^{s(\epsilon,\xi,\delta)})^{-1}(v)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \leq \|v\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}/(1 - C_{\xi,\epsilon}\delta)$

$(\Phi_\xi^{s(\epsilon,\xi,\delta)})^{-1}(\{v \in \mathcal{H}^2(\mathbb{R}^2) : \text{supp}\, v \text{ is compact}\}) \subset \text{Dom}_{+0}(\widetilde{H^\xi})$

since $\Phi_\xi - \Phi_\xi^{s(\epsilon,\xi,\delta)}$ is smooth and has an exponenially decaying property.
By this we can take many elements of the domain.
This is the most important point I get from the work by Mousard.

The operator with the restricted white noise $\xi_R = \sum_{a\in\mathbb{Z}^2\cap\square_R} \chi_a^2\xi$

$$\begin{aligned}\widetilde{H_R^\xi}u = &- \Delta\Phi_{\xi,R}(u) + P_{\xi_R}\Phi_{\xi,R}(u) + \Pi(\Phi_{\xi,R}(u),\xi_R) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi_R)) \\ &+ e^{\Delta}P_u\xi_R + e^{\Delta}{}_uP_{\xi_R}(\Delta^{-loc}\xi_R) + e^{\Delta}P_uY_{\xi,R} + C(u,\xi_R,\xi_R) \\ &+ S(u,\xi_R,\xi_R) + P_{Y_{\xi,R}}u + \Pi(u,Y_{\xi,R}) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}Y_{\xi,R})) \\ &+ P_{\xi_R}(\Delta^{-loc}{}_uP_{\xi_R}(\Delta^{-loc}\xi_R)) + \Pi(\Delta^{-loc}{}_uP_{\xi_R}(\Delta^{-loc}\xi_R),\xi_R) \\ &+ P_{\xi_R}(\Delta^{-loc}P_uY_{\xi,R}) + \Pi(\Delta^{-loc}P_uY_{\xi,R},\xi_R),\end{aligned}$$

$$\Phi_{\xi,R}(u) = u - \Delta^{-loc}P_u\xi_R - \Delta^{-loc}{}_uP_{\xi_R}(\Delta^{-loc}\xi_R) - \Delta^{-loc}P_uY_{\xi,R}$$

$$Y_{\xi,R} = \lim_{\varepsilon\to 0}(\Pi(\Delta^{-loc}\xi_{\varepsilon,R},\xi_{\varepsilon,R}) - \mathbb{E}[\Pi(\Delta^{-loc}\xi_{\varepsilon,R},\xi_{\varepsilon,R})]) \quad (\xi_{\varepsilon,R} = \textstyle\sum_{a\in\mathbb{Z}^2\cap\Lambda_R}\chi_a^2 e^{\varepsilon^2\Delta}\xi)$$

$$\mathrm{Dom}(\widetilde{H_R^\xi}) := \left\{u \in \bigcap_{\epsilon>0}\mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \Phi_{\xi,R}(u) \in \mathcal{H}^2(\mathbb{R}^2)\right\}$$

Properties of the operator with the restricted whitenoise

$$\|\nabla \Phi_{\xi,R}(u)\|^2_{L^2(\mathbb{R}^2)} \leq (u, (\widetilde{H^\xi_R} + k(\xi, R))u)_{L^2(\mathbb{R}^2)}$$

We can show that $\mathrm{Ran}(\widetilde{H^\xi_R} + k(\xi, R)) = L^2(\mathbb{R}^2)$

Lemma (Self-adjointness of the operator with the restricted whitenoise)

The operator $\widetilde{H^\xi_R}$ with the domain $\mathrm{Dom}(\widetilde{H^\xi_R})$ is self-adjoint on $L^2(\mathbb{R}^2)$.

Proof of Theorem on the self-adjointness

For $\forall f \in \mathrm{Ran}(\widetilde{H^{\xi}} + i)^{\perp}$,

$\|f\|^2_{L^2(\mathbb{R}^2)} = \lim_{R\to\infty} (f, \widetilde{\chi_R} f)_{L^2(\mathbb{R}^2)}$, where $\widetilde{\chi_R}$ is C^∞, $= 1$ on $\square_{R-1}$, $= 0$ on $\square_R^c$.

$\widetilde{\chi_R} f = (\widetilde{H^{\xi}_{R+L}} + i)\varphi_{R,L}$ with $\exists \varphi_{R,L} \in \mathrm{Dom}(\widetilde{H^{\xi}_{R+L}})$ and $L > 0$ by the S.A. of $\widetilde{H^{\xi}_{R+L}}$

$\varphi_{R,L}$ is near to $\widetilde{\varphi_{R,L}} := (\Phi_{\xi}^{\boldsymbol{s}(\epsilon,\xi,\delta)})^{-1}(\Phi_{\xi,R+L}^{\boldsymbol{s}(\epsilon,\xi,\delta)}(\varphi_{R,L})) \in \mathrm{Dom}_{+0}(\widetilde{H^{\xi}})$

Since $(\widetilde{H^{\xi}} + i)\widetilde{\varphi_{R,L}} \in \mathrm{Ran}(\widetilde{H^{\xi}} + i)$ is orthogonal to f, we have

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \lim_{R\to\infty} (f, \underbrace{(\widetilde{H^{\xi}_{R+L}} + i)\varphi_{R,L} - (\widetilde{H^{\xi}} + i)\widetilde{\varphi_{R,L}}}_{\downarrow_0 \text{ as } L\to\infty \text{ owing to good estimates of } \Phi_{\xi}^{\boldsymbol{s}}})_{L^2(\mathbb{R}^2)} = 0$$

$\therefore \overline{\mathrm{Ran}(\widetilde{H^{\xi}} + i)} = L^2(\mathbb{R}^2)$

Resolvent convergence

For $\widetilde{H^{\xi}}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

$$\sup_{\|v\|_{L^2(\mathbb{T}^2)}=1} \|(\widetilde{H^{\xi_\varepsilon}} + z)^{-1}v - (\widetilde{H^{\xi}} + z)^{-1}v\|_{L^2(\mathbb{T}^2)} \xrightarrow{\varepsilon\to 0} 0 \text{ for sufficiently large } z \in \mathbb{R}$$

(Allez-Chouk Th.1.6, Mouzard Prop.2.14)

$\lambda_n(\widetilde{H^{\xi_\varepsilon}}) \xrightarrow{\varepsilon\to 0} \lambda_n(\widetilde{H^{\xi}})$ (Allez-Chouk Th.1.6, Mouzard Cor.2.15)

For $\widetilde{H^{\xi}}$ on $\mathbb{R}^2$, (U(2025) Prop.4.1)

$\|(\widetilde{H^{\xi_\varepsilon}} + z)^{-1}v - (\widetilde{H^{\xi}} + z)^{-1}v\|_{L^2(\mathbb{R}^2)} \xrightarrow{\varepsilon\to 0} 0$ for each $v \in L^2(\mathbb{R}^2)$ and $z \in \mathbb{C}\setminus\mathbb{R}$

However the estimate with $\sup_{\|v\|_{L^2(\mathbb{R}^2)}=1}$ may be difficult.

For the identification of the spectrum, we use the method used for stationary random operators.

Proof of Theorem on the spectral identification

For $\forall r \in \mathbb{R}, \varepsilon > 0$ and $L >> 0$,

$$\mathbb{P}(\exists x(\xi) \in \mathbb{Z}^2 \text{ s.t. } |\xi - r| \leq \varepsilon \text{ on } x(\xi) + \Box_L) = 1$$

by the ergodicity of ξ and the Fourier series representation of ξ on $\Box_L$.
$\forall \lambda \in \mathbb{R}, L >> 0, \exists r, c \in \mathbb{R}\ \varphi \in C_0^\infty(x(\xi) + \Box_L) : \|\varphi\| = 1$
s.t. $\|(\widetilde{H^\xi} - \lambda) \underset{\underset{(\Phi^{s(\epsilon,\xi,L,\delta,x(\xi))}_{\xi,L,x(\xi)})^{-1}(\varphi)}{\|}}{\widetilde{\varphi}} \|_{L^2(\mathbb{R}^2)} < c\varepsilon$.

$\therefore$ We can take a Weyl sequence. $\therefore \lambda \in \mathrm{Spec}(\widetilde{H^\xi})$,
$\therefore \mathrm{Spec}(\widetilde{H^\xi}) = \mathbb{R}$.